Introduction: Grothendieck Topologies, Sheaves, and Cohomology

(10 exercises = A, must present the proof, also one project = A).

Project 1: Prove $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$.

X a top. space. Consider the category C/X of open subsets of X, with morphisms only the inclusions. A presheaf of sets (or anything else) is a contravariant functor $F: C/X \to \text{Sets}$. So for each $U \subset X$, we get a set $F(U)$ and a "restriction" map:

$$\begin{array}{ccc} U & \subset & V \\ \downarrow & & \downarrow \\ F(U) & \xleftarrow{r_{UV}} & F(V) \end{array}$$

Set $P(C/X)$ to be the category of presheaves of abelian groups on X. Since Ab is abelian, so is $P(C/X)$. Now let $S(C/X)$ be the category of sheaves of Ab. grps, and is defined as a full subcategory of $P(C/X)$. We need a notion of a covering, i.e. a collection $\{U_i\}$ of open subsets with $\bigcup U_i = U$. Then we have the sheaf axiom: $S \in P(C/X)$ is a sheaf if for any covering $\{U_i \to U\}$, and fiber product $U_i \cap U_j$, the sequence

$$S(U) \longrightarrow \prod_i S(U_i) \rightrightarrows \prod_{i,j} S(U_i \times_U U_j)$$

Then $S(U) \to \prod_i S(U_i)$ is the equalizer of the diagram.

Now note $S(C/X)$ is an abelian category with two natural functors. The inclusion (forgetful) $\underbrace{F: S(C/X) \to P(C/X)}_{\text{left exact}}$ and sheafification $+: \underbrace{P(C/X) \to S(C/X)}_{\text{exact}}$. These are actually adjoint!

Since $S(C/X)$ has enough injectives, we form an injective resolution $S \to I^0 \to I^1 \to \cdots$ and take global sections. Note an exact sequence of sheaves is not the same as an exact sequence of presheaves.

Def: Let C be a category with fiber products. A Grothendieck topology on C is the following data: a collection of coverings $\{U_i \to U\}_{i \in I}$ s.t.

1) Any isomorphism $U \xrightarrow{\sim} U$ is a covering.
2) If $\{U_i \to U\}_I$ and $\{V_{ij} \to U_i\}_J$ is a covering $\forall i$, then $\{V_{ij} \to U\}$ is a covering.
3) If $\{U_i \to U\}_I$ is a covering, for all $V \to U$, $\{U_i \times_U V \to V\}$ is a covering.

Fact: We get abelian categories $P(C) \underset{F}{\overset{+}{\rightleftarrows}} S(C)$, $S(C)$ has enough injectives, and for all $X \in C$, we get $H^i(X, S)$, done in the same way!

Def: A category with a Grothendieck topology is a site.

Example of a set-valued pre-sheaf on any category:
$V \in C$, define $h_V: C^{op} \to \text{Sets}$ by $h_V(X) = \text{Hom}_C(X, V)$.

Def: Let C be a category with fiber products. Then a Grothendieck topology E on C is said to be canonical if it is the finest one in which every h_V, $V \in C$ is a sheaf.

For us, C = Schemes, and we have 3 topologies: Zar, Étale, Flat. For Zar, $(U_i \xrightarrow{f_i} U)$ is a covering if $\forall i$, $f_i : U_i \to U$ is an open embedding, and $\bigcup f_i(U_i) = U$. Now flat descent $\Rightarrow$ Flat is sub canonical $\Rightarrow$ Zar, Étale is sub canonical.

Now in Zar, fix $h_V : Sch^{op} \to Sets$, and choose $(U_i \to U)$. Then:

$$\mathrm{Hom}(U,V) = h_V(U) \to \prod h_V(U_i) \rightrightarrows \prod h_V(U_i \cap U_j)$$

to be an equalizer, $\forall i$, $(U_i \xrightarrow{\alpha_i} V)$, $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$.

For which Schemes V will h_V take values in Ab? The answer relies on the notion of group schemes.

Def: A group scheme $G/X \in Sch/X$ is a morphism $G \xrightarrow{\pi} X$ and X-morphisms $\mu : G \times_X G \to G$ and $\beta : G \to G$ and $\underbrace{e : X \to G}_{\text{section}}$ satisfying:

1) (Associativity)

$$\begin{tikzcd} G \times G \times G \arrow[r, "1 \times \mu"] \arrow[d, "\mu \times 1"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd}$$

Commutes.

2) Both compositions $G \xrightarrow{\Delta} G \times G \underset{\beta \times 1}{\overset{1 \times \beta}{\rightrightarrows}} G \times G \xrightarrow{\mu} G$ are equal to $e \circ \pi : G \to G$.

3) Law of identity: The compositions

$$\begin{tikzcd} & X \times_X G \arrow[dr, "e \times id"] & & \\ G \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & G \times_X G \arrow[r, "\mu"] & G \\ & G \times_X X \arrow[ur, "id \times e"'] & & \end{tikzcd}$$

are equal to id_G.

4) G is commutative if $\mu = \mu \circ i$, where $i : G \times G \to G \times G$ swaps factors.

Exercise 0: Show $G \in Sch/X$ is a group scheme iff the functor h_G takes values in Grp.

Def: A group scheme G/X acts on Y/X if there is $\sigma : G \times_X Y \to Y$ s.t.

1) $$\begin{tikzcd} G \times G \times Y \arrow[r, "1 \times \sigma"] \arrow[d, "\mu \times 1"'] & G \times Y \arrow[d, "\sigma"] \\ G \times Y \arrow[r, "\sigma"] & Y \end{tikzcd}$$ commutes

2) the composition $Y = X \times_X Y \xrightarrow{e \times 1} G \times Y \xrightarrow{\sigma} Y$ is id_Y.

Exercise 1/2: Prove that a group scheme G/X acts on Y/X iff the group functor h_G acts on the set functor h_Y. (Note this allows us to work w/ points!)

Yoneda Lemma: The functor $h: C \to P(C) = \mathrm{Fun}(C^{op}, \mathrm{Sets})$ is fully faithful.

Ex 1: $G_a = \mathrm{Spec}\,\mathbb{Z}[x] \in Sch/_{\mathrm{Spec}\,\mathbb{Z}}$. Define $\mu: G_a \times G_a \to G_a$ as $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x] \otimes \mathbb{Z}[x]$ via $x \longmapsto x \otimes 1 + 1 \otimes x$, $\beta: G_a \to G_a$ by $x \mapsto -x$, and $e: \mathrm{Spec}\,\mathbb{Z} \to G_a$ by $x \mapsto 0$. (commutative).

So then $h_{G_a}(X) = \mathrm{Hom}_{Sch}(X, G_a) = \mathrm{Hom}_{Ring}(\mathbb{Z}[x], \Gamma(X, \mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)$.

Ex 2: $G_m = \mathrm{Spec}\,\mathbb{Z}[x, x^{-1}]$. Then:

$$\left.\begin{aligned}
\mu: \mathbb{Z}[x,x^{-1}] &\longrightarrow \mathbb{Z}[x,x^{-1}] \otimes \mathbb{Z}[x,x^{-1}] \\
x &\longmapsto x \otimes x \\
\beta: \mathbb{Z}[x,x^{-1}] &\longrightarrow \mathbb{Z}[x,x^{-1}] \\
x &\longmapsto x^{-1} \\
e: \mathbb{Z}[x,x^{-1}] &\longrightarrow \mathbb{Z} \\
x &\longmapsto 1.
\end{aligned}\right\}\ \text{Actually a commutative grp. scheme.}$$

Note $h_{G_m}(X) = \mathrm{Hom}_{Sch}(X, G_m) = \mathrm{Hom}_{Ring}(\mathbb{Z}[x, x^{-1}], \Gamma(X, \mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)^{*}$.

Ex 3: $GL_n = \mathrm{Spec}\,\mathbb{Z}[x_{ij}][\det(x_{ij})^{-1}]_{1 \le i,j \le n}$.

$$\mu \text{ sends } \quad (x_{ij}) \longmapsto \sum x_{ik} \otimes x_{kj}$$
$$\det \longmapsto \det \otimes \det.$$

An abelian category is called Grothendieck if it satisfies a list of conditions (look this up). A theorem of Grothendieck says that any such category has enough injectives (google this for a proof). Many sites are Grothendieck categories.

Lets take another look at cohomology. For any category C, we have the constant presheaf $\mathbb{Z}_C$ sending any object to $\mathbb{Z}$ and morphism to $id_{\mathbb{Z}}$.

Lemma: Suppose C has a final object X. Then the two functors $P(C) \longrightarrow Ab$ given by evaluation at X: $F \mapsto F(X)$ and the functor $h_{\mathbb{Z}_C}(F) = \mathrm{Hom}_{P(C)}(\mathbb{Z}_C, F)$ are isomorphic.

Proof: Suppose we have $\varphi \in \mathrm{Hom}_{P(C)}(\mathbb{Z}_C, F)$. Then we define the image of 1 under $\mathbb{Z}(X) \to F(X)$ to be the element of $F(X)$. Conversely given $\alpha \in F(X)$ choose $\varphi \in \mathrm{Hom}_{P(C)}(\mathbb{Z}_C, F)$ by the map $\varphi_X(1) = \alpha$, $\varphi_X: \mathbb{Z}(X) \to F(X)$. One can check this is well defined as X is final. ∎

Now by the adunction $P \underset{F}{\overset{+}{\rightleftarrows}} S$, $\mathrm{Hom}(\mathbb{Z}_C, F) = \mathrm{Hom}(\mathbb{Z}_C^{+}, F)$. Since the left one is just $F(X)$, we now have two left exact functors. Thus we have

$$H^i(X, F) = \mathrm{Ext}^i(\mathbb{Z}, F).$$

Lemma: Suppose (C, E) is a trivial site (only covering is isomorphism). Then $S(C_E) = P(C_E)$.

Proof: Consider the covering $U \xrightarrow[\sim]{\alpha} U$. Then in the fiber product:

$$\begin{tikzcd} U \arrow[r, "\beta = \alpha^{-1}"] \arrow[d, "\beta"'] & U \arrow[d, "\alpha"] \\ U \arrow[r, "\alpha"] & U \end{tikzcd}$$

Then $F(U) \xrightarrow{\alpha^*} F(U) \overset{\beta^*}{\underset{\beta^*}{\rightrightarrows}} F(U)$ is automatic. ∎

Ex: Let G be a group, considered as a category with trivial topology (indeed, all morphisms are coverings). Then a pre-sheaf on G is a sheaf, and is a G-module.

Then $M \to M^G$ is left exact, and so can be given cohomology. Since $M^G = \mathrm{Hom}(\mathbb{Z}, M)$, so $H^i(G, M) = \mathrm{Ext}^i(\mathbb{Z}, M)$ – the group cohomology! Now $H^i(G, F) = \mathrm{Ext}^i(\mathbb{Z}_G, F)$.

Since we don't have a final object, $\mathrm{Hom}(\mathbb{Z}_G, F) \neq F(\cdot)$ indeed $F(\cdot) = M$, so there would be no cohomology! Ext is "better".

Flat Morphisms

All rings are noetherian. All schemes are locally noetherian.

Def: Let A be a ring. An A-module is flat if $- \otimes_A M$ is an exact functor.

In particular any free A-module is flat, and any localization of a flat module is flat. A theorem of commutative algebra tells us its enough to use finitely generated modules, so we make that assumption.

Def: A map of rings $f: A \to B$ is flat if B is a flat A-module.

Prop: $f: A \to B$ is flat if for every ideal $I \subset A$, the map $I \otimes_A B \to A \otimes_A B = B$ is injective.

Proof: Let $0 \to M' \xrightarrow{g} M$ be exact with f.g. A-modules.

Case (a): M is free of rank r. Then $r = 1$ is just $0 \to I \to A$, which is the assumption. If $r > 1$, write $M = M_1 \oplus M_2$ – free of lower rank. $\otimes B$ gives:

$$\begin{tikzcd} 0 \arrow[r] & M_1 \otimes B \arrow[r] & M \otimes B \arrow[r, "p"] & M_2 \otimes B \arrow[r] & 0 \\ & g^{-1}(M_1) \otimes B \arrow[r] \arrow[u] & M' \otimes B \arrow[r] \arrow[u, "g \otimes B"] & pg(M') \otimes B \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

a diagram chase shows the claim.

Case (b): M arbitrary. Let $x_1, \dots, x_r$ generate M. Then we get, after tensoring with B:

$$\begin{tikzcd} N \arrow[r] & \bigoplus A x_i \arrow[r] & M \arrow[r] & 0 \\ N \arrow[r] \arrow[u, equal] & h^{-1} g(M') \arrow[r] \arrow[u, hook] & M' \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

and again a diagram chase is enough by (a). ∎

Prop: If $f: A \to B$ is flat, so is $S^{-1}A \to T^{-1}B$ if $f(S) \subset T$. Conversely if for all maximal ideals $m \subseteq B$, $A_{f^{-1}(m)} \to B_m$ is flat, then f is flat.

In particular, we get $f: A \to B$, $f^*: \operatorname{Spec} B \to \operatorname{Spec} A$, and this map of schemes is flat if it's flat at all closed points.

Def: A map $f: X \to Y$ of schemes is flat if $\forall y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.

Exercise 1: Show f is flat iff $\forall$ affine open $V \subset Y$, $U \subset X$ with $f(V) \subset U$, the morphism $\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$ is flat.

Note:
1) Composition of flat morphisms is flat. Clear from commutative algebra.
2) Any open immersion is flat. Take $U \xrightarrow{f} X$, then $\mathcal{O}_{U,p} = \mathcal{O}_{X,f(p)}$, and the identity is flat.
3) Any base change of a flat morphism is flat. Indeed we can assume affine, so:

$$\begin{array}{ccc} \operatorname{Spec}(C \otimes_A B) & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow f \text{ - flat} \\ \operatorname{Spec} C & \longrightarrow & \operatorname{Spec} A \end{array}$$

we get

$$\begin{array}{ccc} C \otimes_A B & \longleftarrow & B \\ \uparrow & & \uparrow f^* \text{ - flat by Ex. 1.} \\ C & \longleftarrow & A \end{array}$$

Some theorems:

Thm (Hartshorne III.9.5): Let $f: X \to Y$ be a flat morphism of schemes of finite type over a field k. Then for all $x \in X$, $y = f(x)$, then $\dim_x X = \dim_x (X_y) + \dim_y Y$.

Thm (Hartshorne III.9.9): Let T be an integral scheme and consider $\mathbb{P}^n_T$. We have a closed subscheme $X \subset \mathbb{P}^n_T$ over T. For each $t \in T$, $P_t \in \mathbb{Q}[z]$ is the Hilbert polynomial of $X_t \subset \mathbb{P}^n_{k(t)}$. Then $f: X \to T$ is flat iff P_t is independent of t.

In particular, all fibers have the same dimension.

Def: Let $f: A \to B$ be a flat morphism of rings. Then f is faithfully flat if for any A-module $M \neq 0$, $M \otimes_A B \neq 0$.

In particular, a faithfully flat morphism is injective. Let $I = \ker f$, then we have:

$$\left.\begin{array}{ccccccc} 0 & \to & I & \to & A & & \text{exact} \\ \| & & \downarrow 1 \otimes i & & \downarrow f & & \\ 0 & \to & I \otimes_A B & \to & A \otimes_A B = B & & \text{exact.} \end{array}\right\} \begin{array}{l} \text{Since } I \otimes_A B \to B \text{ is injective, we see} \\ \text{that } I \otimes_A B = 0 \Rightarrow I = 0. \end{array}$$

Prop: Let $A \neq 0$, $f: A \to B$ flat. Then TFAE:
i) f is faith. flat.
ii) $M' \to M \to M''$ is exact iff $B \otimes M' \to B \otimes M \to B \otimes M''$ is an exact.
iii) $f^{\#}: \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.
iv) $\forall$ max. ideals $m \subset A$, $f(m) \cdot B \neq B$. In particular, a flat local hom of local rings is faith. flat always.

Proof: See proposition 2.7 in Milne's "Étale Cohomology" ∎

We now define a fundamental object, the flat topology, after giving some basics.

Def: A morphism $f: Y \to X$ is faith. flat if it is flat and surjective.

It is worth remarking that faith. flat morphisms are preserved under composition and base change.

Def: We define a Grothendieck topology on Sch by for all schemes X, a covering set $\operatorname{Cov}(X) = \{U_i \to X\}$, where $\coprod U_i \to X$ is faith. flat.

Corollary: Let $f: Y \to X$ be flat, $y \in Y$, and $x = f(y)$. Let $x \in \overline{\{x'\}}$. Then there is a $y' \in Y$ such that $f(y') = x'$ and $y \in \overline{\{y'\}}$.

Proof: Corollary 2.8 in Étale Cohomology (EC). ∎

Thm: Any flat morphism that's locally of finite type is open.

Note locally of finite type. For example $\mathbb{Z} \xrightarrow{\text{flat}} \mathbb{Z}_{(2)}$ gives $\operatorname{Spec} \mathbb{Z} \leftarrow \operatorname{Spec} \mathbb{Z}_{(2)}$, which not of finite type. Then the image of $\operatorname{Spec} \mathbb{Z}_{(2)}$ is not open.

Proof: There is an open cover $\operatorname{Spec} A_i$ of X and $f^{-1}(U_i) = \operatorname{Spec} B_{ij} = V_{ij}$, with each B_{ij} is a f.g. A_i-algebra. Moreover, $f|_{V_{ij}}: V_{ij} \to U_i$ is flat, so its enough to show this is open. Thus we can assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with B a f.g. A-algebra.

We know $\operatorname{Im}(f)$ is closed under generalization. Since the image of constructible sets is constructible and constructible sets closed under generalization are open, we are done. ∎

Exercise 2: Prove Exercise II.3.18 (a)-(d) in Hartshorne.

Exercise 3: Prove Exercise II.3.19 (a)-(d) in Hartshorne.

Exercise 4: Prove that a finite morphism is a closed morphism.

Consider a finite flat morphism $f: Y \to X$. Then $f(Y)$ is open and closed in X (hence surjective if X is connected).

Thm: Let M be a f.g. A-module (A noetherian). TFAE:

i) M is flat as an A-module;
ii) M_m is a free A_m-module, for all maximal ideals $m \subset A$;
iii) $\tilde{M}$ is a locally free $\mathcal{O}_{\mathrm{Spec}\,A}$-module;
iv) M is projective;
v) If A is also integral, then $\dim_{k(p)}(M \otimes_A k(p))$ is constant on Spec A.

Proof: Thm 2.9 in EC.

(ii) $\Rightarrow$ (iii). Fix a max. ideal $m \subset A$. Then M_m is a free A_m-module (w/ finite basis). Hence we have a map $M \to M_m$, and since

$$M/mM \xrightarrow{\sim} M_m/mM_m,$$

so choose $x_1, \ldots, x_n \in M$ which are a basis for M_m. We then have a map: $\oplus Ax_i \to M$ and hence an isomorphism $(\oplus Ax_i)_m \xrightarrow{\sim} M_m$ and hence we have an open set containing $m \in \mathrm{Spec}\,A$ upon which $\tilde{M}|_U \cong \oplus \mathcal{O}_U$. ∎

Exercise 5: Prove (i) $\Rightarrow$ (iv), (i) $\Rightarrow$ (ii), (iv) $\Leftrightarrow$ (v) (Ex. II.5.8 in Hartshorne).

Example: $f: Y \to X$ be flat and finite. Then $f_* \mathcal{O}_Y$ is coherent on X, which is flat over $\mathcal{O}_X$. Then we have from the above that its locally free.